Asymptotic Density in Generic Extensions

Jan Grebík
Charles University in Prague

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Definition

For $A \subseteq \mathbb{N}$ we define upper and lower asymptotic density of A by this formulas

$$d^*(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}$$

$$d_*(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

Let $\mathcal{D} = \{A : d^*(A) = d_*(A)\}$. We define asymptotic density for $A \in \mathcal{D}$ by

$$d(A) = \lim_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

$\mathcal{D} \subseteq \mathcal{P}(\mathbb{N})$ but there are sets which does not have asymptotic density.

Example

For $i \in \mathbb{N}$ let $A_i = \{n : \textit{first digit of } n \textit{ is } i\}$ then $A_i \notin \mathcal{D}$ because $d_*(A_i) = \frac{1}{9i}$ and $d^*(A_i) = \frac{10}{9(i+1)}$.

In fact $\mathcal{D}$ is not even a subalgebra of $\mathcal{P}(\mathbb{N})$ because it is not closed under unions.

Fact

If $A, B \in \mathcal{D}$ and $A \cap B = \emptyset$ then $A \cup B \in \mathcal{D}$ and $d(A \cup B) = d(A) + d(B)$.

Even more there exists a measure $\overline{d}$ on $\mathcal{P}(\mathbb{N})$ with $\overline{d} \restriction_{\mathcal{D}} = d$.

Proposition

Let $s = \{a_i : i \in \mathbb{N}\}$ be a sequence of natural numbers such that for each $i \neq j$ greatest common divisor $gcd(a_i, a_j) = 1$. Then the set

$$A_s = \{n : \forall i \in \mathbb{N}\ a_i \nmid n\}$$

has asymptotic density and

$$d(A_s) = \prod_{i \in \mathbb{N}} (1 - \frac{1}{a_i}).$$

Example1

For $k > 1$ let $P_k = \{n : \forall p\ p^k \not| n\}$ then $P_k = A_{s_k}$ where $s_k = \{p_i^k\}$ where p_i is the i-th prime number. Then by Euler formula

$$d(P_k) = \prod_{i \in \mathbb{N}} (1 - \frac{1}{p_i^k}) = \frac{1}{\sum_{n \in \mathbb{N}} \frac{1}{n^k}} = \frac{1}{\zeta(k)},$$

where ζ is Riemann zeta function.

Example2

P_2 is called set of squarefree numbers and $d(P_2) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$.
Let $\alpha \in \{0, 1\}$ and $P_2^\alpha = \{k : k = p_{i_1} ... p_{i_m}\ and\ m \equiv \alpha\ mod\ 2\}$ then

$$d(P_2^\alpha) = \frac{3}{\pi^2}.$$

Definition

$$\mathcal{Z} = \{A : A \subseteq \mathbb{N}, d(A) = 0\}$$

$$\mathcal{F}in = \{A : A \subseteq \mathbb{N}, |A| < \omega\}$$

Theorem (???)

$$\mathcal{P}(\mathbb{N})/\mathcal{Z} \cong \mathcal{P}(\mathbb{N})/\mathcal{F}in * \mathbb{B}(\mathfrak{c})$$

Proof, sketch of first part

Let $I_n = [2^n, 2^{n+1})$ and $X \subseteq \mathbb{N}$. Define $f : \mathcal{P}(\mathbb{N})/\mathcal{F}in \to \mathcal{P}(\mathbb{N})/\mathcal{Z}$ with following formula

$$f([X]_{\mathcal{F}in}) = [\bigcup_{n \in X} I_n]_{\mathcal{Z}}.$$

Then since

$$X \in \mathcal{Z} \Leftrightarrow \lim_{n \to \infty} \frac{X \cap [2^n, 2^{n+1})}{2^n} = 0$$

f is regular embedding.

First question about generic extension is size of $\mathfrak{c}$.

Proposition

Forcing with $\mathcal{P}(\mathbb{N})/\mathcal{F}in$ collapses $\mathfrak{c}$ to $\mathfrak{h}$, where $\mathfrak{h}$ is distributivity number of $\mathcal{P}(\mathbb{N})/\mathcal{F}in$.
$\mathbb{B}(\kappa)$ is c.c.c. so does not collaps cardinals.
All together $\mathfrak{c}$ in extension is $\mathfrak{h}$ in groundmodel.

An ultrafilter $\mathcal{U}$ is selective if for every partition $\{J_n\}_{n\in\omega}$ of $\mathbb{N}$ there exists $k \in \omega$ such that $J_k \in \mathcal{U}$ or there exist $U \in \mathcal{U}$ such that $|U \cap J_n| = 1$ for every $n \in \omega$.

Question

Is there a selective ultrafilter in extension by $\mathcal{P}(\mathbb{N})/\mathcal{Z}$??

Answer

If $\mathfrak{h} = \omega_1$ in V then we can construct a selective ultrafilter because CH is true in $V[G]$.
Otherwise we do not know.

Fact

Because of $\mathbb{B}(\mathfrak{c})$ $cov(\mathcal{M}) = non(\mathcal{N}) = \omega_1$ in extension.

Theorem (Canjar)

Every filter generated by $< \mathfrak{c}$ elements can be extended to a selective ultrafilter iff $cov(\mathcal{M}) = \mathfrak{c}$.

Generic filter over $\mathcal{P}(\mathbb{N})/\mathcal{F}in$ is selective in extension.

Theorem (Kunen)

Selective ultrafilter in V cannot be extended to P-(selective) ultrafilter in $V[G]$ where G is generic over $\mathbb{B}(\kappa)$.
Forcing with $\mathbb{B}(\kappa)$ where $\kappa > \mathfrak{c}$ creates universe without selective ultrafilters.

What about $\mathbb{B}(\kappa)$ where $\kappa \leq \mathfrak{c}$??

Proposition

In $V[G]$ $\mathfrak{h} = \omega_1$ and because of that

$$V^{\mathcal{P}(\mathbb{N})/\mathcal{Z} * \mathcal{P}(\mathbb{N})/\mathcal{Z}} \vdash CH.$$

In otherwords there are selective ultrafiltres after two step iteration of $\mathcal{P}(\mathbb{N})/\mathcal{Z}$.

Thank you.